NORMAL FUNCTIONS AND A CLASS OF ASSOCIATED BOUNDARY FUNCTIONS

BY

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ABSTRACT

Let μ' be the family of non-empty closed subsets of the Riemann sphere and Λ the family of continuous curves λ with values in the unit disk and $\lim_{t\to 1} |\lambda(t)| = 1$. A meromorphic function f in $|z| < 1$ induces a mapping $\hat{f}$ from Λ into μ' by setting $\hat{f}(\lambda)$ equal to the cluster set of f on λ. The authors show that if $\hat{f}$ is continuous then existence of an asymptotic value at $e^{i\theta}$ implies the existence of an angular limit. Further if the spherical derivative of f is $o(1/(1-|z|))$ then $\hat{f}$ is constant on every open disk in the space Λ.

1. **Introduction and notation.** Let $D = \{z \mid |z| < 1\}$ denote the unit disk and $C = \{z \mid |z| = 1\}$ its boundary. For points z_1 and z_2 in D the non-Euclidean (hyperbolic) distance between z_1 and z_2 is given by the formula

$$\rho(z_1, z_2) = \frac{1}{2} \log \frac{|\bar{z}_1 z_2 - 1| + |z_1 - z_2|}{|\bar{z}_1 z_2 - 1| - |z_1 - z_2|}$$

We designate the extended complex plane by W and the chordal distance between w_1 and w_2 in W by

$$\chi(w_1, w_2) = \frac{|w_1 - w_2|}{\sqrt{(1+|w_1|^2)}\,\sqrt{(1+|w_2|^2)}}$$

Let u' denote the family of non-empty closed subsets of W with the standard Hausdorff topology generated by χ [4, pp 20–32], where the distance between two sets $A, B \in u'$ will be denoted by dist (A, B). The set Λ will be the family of all continuous curves $\lambda(t)$ in D with $\lambda(0) = 0$ and $\lim_{t\to 1}|\lambda(t)| = 1$. The symbol $\Lambda^*(\theta)$ indicates the subset of curves of Λ which approach $e^{i\theta}$ nontangentially, i.e., $\lambda(t) \in \Lambda^*(\theta)$ if $\limsup_{t\to 1} |\arg(z(t) - e^{i\theta}) - \theta| < \pi/2$. The cluster set of a complex-valued function f along the path $\lambda(t)$ in D terminating in C is defined as follows

$$C_\lambda(f) = \{w \mid \text{there is } \{z_n\}, \quad z_n \in \lambda$$

$$\lim_{n\to\infty} |z_n| = 1 \text{ with } \lim_{n\to\infty} f(z_n) = w\}.$$

Received February 15, 1966, and in revised form, July 21, 1966.

In this paper we define a metric $\hat{\rho}$ on Λ and show that with this metric topology $\Lambda^*(\theta)$ is an arcwise connected Hausdorff subspace of Λ. There is the usual geometric interpretation of ε-spheres in this metric topology. That is two Jordan curves $\lambda_1(t)$ and $\lambda_2(t)$ in Λ lie in the same ε-sphere if the curve $\lambda_2(t)$ lies in the non-Euclidean ε-"envelope" about $\lambda_1(t)$ and if $\lambda_1(t)$ lies in the non-Euclidean ε-"envelope" about λ_2.

We shall need the following definitions and results.

DEFINITION 1. A function f defined in D withl vaues in! W is said to e normal if and only if whenever $\{S_\alpha(z)\}$ denotes the family of 1:1 conformal mappings of D onto D, the family $\{f(S_\alpha(Z))\}$ is normal in the sense of Montel. For meromorphic functions this definition is due to Lehto and Virtanen [6, p. 53]. Each function f in D determines a natural map $\hat{f}$ of the space Λ into the space μ'. This map is defined as follows

$$\hat{f}(\lambda) = C_\lambda(f)$$

It is shown in §3 that for a continuous normal function f, $\hat{f}$ is a continuous function.

Lehto and Virtanen [6, pp 59–62] have shown that if a meromorphic function f is normal and has asymptotic value α at $e^{i\theta}$ then f has angular limit α at $e^{i\theta}$.

DEFINITION 2. A continuous function f mapping D into W is said to have the Lindelöf property at $e^{i\theta}$ if whenever f has asymptotic value α at $e^{i\theta}$ then f has angular limit α at $e^{i\theta}$.

Using the results of Lehto and Virtanen we will prove the following theorem;

THEOREM. *If f is meromorphic and $\hat{f}$ is continuous then f has the Lindelöf property at each $e^{i\theta}$.*

Finally, in 4 it is shown that if $\rho(f)(z) = o(1/1-|z|)$ where $\rho(f)$ denotes the spherical derivative of f then $\hat{f}$ is a constant value on every open disk in Λ.

2. The ρ^* function.

Bagemihl and Seidel [2, p. 263] have used the non-Euclidean Fréchet distance to define a metric on the family of boundary paths in D. However, this metric is defined in terms of topological correspondences between the two given boundary paths. The $\hat{\rho}$ function is patterned after that of the metric function used in the Hausdorff topology with the non-Euclidean metric as the defining tool. For any set $A \subset D$ and any point $z \in D$ set

$$\rho(z, A) = \underset{y \in A}{\text{g.l.b.}}\ \rho(z, y).$$

LEMMA 1. *The function (possibly infinite-valued)*

$$\rho^*(\lambda_1, \lambda_2) = \max\left(\sup_{x \in \lambda_1} \rho(x, \lambda_2),\ \sup_{y \in \lambda_2} \rho(y, \lambda_1)\right)$$

satisfies the metric properties for any three curves $\lambda_1, \lambda_2, \lambda_3$ *such that* $\rho^*(\lambda_2, \lambda_3)$ *and* $\rho^*(\lambda_1, \lambda_2)$ *are finite.*

Proof. (This is the standard proof which we give for the sake of completeness only.) If $\rho^*(\lambda_1, \lambda_2) = 0$ then $\rho(x, \lambda_2) = 0$ for each $x \in \lambda_1$.

Since there is a point $y = y(x) \in \lambda_2$ with $\rho(, \lambda_2) = \rho(x, y(x))$ we have $\lambda_1 \subseteq \lambda_2$. The reverse inclusion is similarly verified so that $\lambda_1 = \lambda_2$. The symmetry is clear.

If $\rho^*(\lambda_2, \lambda_3)$ and $\rho^*(\lambda_1, \lambda_2)$ are finite we show that $\rho^*(\lambda_1, \lambda_3) \leqq \rho^*(\lambda_1, \lambda_2) + \rho^*(\lambda_2, \lambda_3)$. For if $x \in \lambda_1, y \in \lambda_2, z \in \lambda_3$ then

$$\rho(x, z) \leqq \rho(x, y) + \rho(y, z). \tag{2.0}$$

Assume $\rho^*(\lambda_1, \lambda_3) = \sup_{x \in \lambda_1} \rho(x, \lambda_3)$. Taking the greatest lower bound of both sides of (2.0) for $z \in \lambda_3$ we obtain

$$\rho(x, \lambda_3) \leqq \rho(x, y) + \rho(y, \lambda_3). \tag{2.1}$$

Now for $x \in \lambda_1$ let $y = y(x)$ be a point of λ_2 such that

$$\rho(x, y(x)) = \rho(x, \lambda_2). \tag{2.2}$$

Combining (2.1) and (2.2)

$$\sup_{x \in \lambda_1} \rho(x, \lambda_3) \leqq \sup_{x \in \lambda_1} \rho(x, \lambda_2) + \sup_{x \in \lambda_1} \rho(y(x), \lambda_3).$$

Thus

$$\begin{aligned} \rho^*(\lambda_1, \lambda_3) &\leqq \sup_{x \in \lambda_1} \rho(x, \lambda_2) + \sup_{x \in \lambda_1} \rho(y, \lambda_3) \\ &\leqq \rho^*(\lambda_1, \lambda_2) + \rho^*(\lambda_2, \lambda_3). \end{aligned}$$

If $\rho^*(\lambda_1, \lambda_3) = \sup_{z \in \lambda_3} \rho(z, \lambda_1)$ a similar argument gives also the above inequality.

It is convenient to define a metric for Λ in the usual fashion. For $\lambda_1, \lambda_2 \in \Lambda$ let

$$\hat{\rho}(\lambda_1, \lambda_2) = \left\{ \begin{array}{ll} \dfrac{\rho^*(\lambda_1, \lambda_2)}{1 + \rho^*(\lambda_1, \lambda_2)} & , \text{if } \rho^*(\lambda_1, \lambda_2) < +\infty; \\ 1 & , \text{if } \rho^*(\lambda_1, \lambda_2) = +\infty; \end{array} \right\}$$

then $\hat{\rho}$ is a metric for Λ. It is only necessary to observe that the inequalities of Lemma 1 show that if $\rho^*(\lambda_1, \lambda_2)$ and $\rho^*(\lambda_2, \lambda_3)$ are finite then $\rho^*(\lambda_1, \lambda_2)$ is also finite and the triangle inequality is valid. If $\rho^*(\lambda_1, \lambda_2) = +\infty$ then at least one of $\rho^*(\lambda_1, \lambda_3)$ or $\rho^*(\lambda_3, \lambda_2)$ also equals infinity.

We remark that if α is the radius terminating at $e^{i\theta}$ then the set of curves λ such that $\hat{\rho}(\alpha, \lambda) < 1$ is just $\Lambda^*(\theta)$.

We prove that $\Lambda^*(\theta)$ is an arcwise connected space in the $\hat{\rho}$ metric. For notational clarity and without loss of generality we prove this result in the case in which

$\theta = 0$. In order to prove the theorem we utilize a distinguished class of points in $\Lambda^*(0)$. Let $H(\beta)$ be the hypercycle joining $+1$ to -1 and making angle $(-\pi/2 < \beta < \pi/2)$ with the diameter $\alpha = \operatorname{Im}(z) = 0$. For interior points z^* of α, since $H(\beta)$ is parallel to α, the non-Euclidean distance of the hypercycle $H(\beta)$ to z^* is given by

$$M = \frac{1}{2}\log\cot\left(\frac{\pi}{4} - \frac{\beta}{2}\right). \tag{1}$$

THEOREM 1. *Each subspace $\Lambda^*(\theta)$ is arcwise connected.*

Proof. Clearly we need only prove the case $\Lambda(0)$. It suffices to show that each $\lambda(t) \in \Lambda^*(0)$ can be continuously deformed in the $\hat{\rho}$ metric to the diameter α. To this end let $\lambda(t)$ be given. There is a number $M = M(\lambda)$ such that $\lambda(t)$ is contained in the symmetric Stolz domain formed at $z = 1$ by hypercycles $H(\beta)$ and $H(-\beta)$ where β is the solution of (1) for $M(\lambda)$. For each $z \in D$, let F_z denote the non-Euclidean straight line through z perpendicular to α. If we denote by M_r the non-Euclidean distance of the hypercycles $H(r\beta)$ from α then it is clear that M_r tends to zero as r tends to zero. Now if $z' = \lambda(t)$ is a point of λ, $\operatorname{Im} z' > 0$, $H(r\beta)$ is a hypercycle and if $\rho(z'; \alpha) \geqq M_r$ then define the projection of z' on $H(r\beta)$ to be the unique point $\xi \in H(r\beta) \cap F_z$. For points z' with $\operatorname{Im} z' < 0$ we make a similar arrangement.

Define the map σ of $(0,1)$ into $\Lambda^*(0)$ as follows: $\sigma(r) = \lambda_r(t)$ where

$$\lambda_r(t) = \begin{cases} \lambda(t) = z \text{ if } \rho(z,\alpha) < M_r; \\ \xi = \text{projection } \lambda(t) \text{ on } H(r\beta) \text{ if} \\ \qquad \rho(\lambda(t),\alpha) \geqq M_r. \end{cases}$$

If $r_0 \in (0,1)$ then $\rho(x, \lambda_{r_0}(t)) \leqq |M_r - M_{r_0}|$ for $x \in \lambda_r$. Thus $\rho^*(\lambda_r, \lambda_{r_0})$ tends to zero as r tends to r_0 and the theorem is proven. We might remark that one could show in a similar manner that given any $\lambda_1 \in \Lambda$ the subspaces for which $\hat{\rho}(\lambda_1, \lambda) < 1$ are all arcwise connected. For, if $\hat{\rho}(\lambda_1, \lambda) < d < 1$, then consider the envelop about λ_1 formed by disks of non-Euclidean radii r, $0 \leqq r \leqq d$, with centers on λ_1. Now λ is contained in this envelop. Let λ_r and $\bar{\lambda}_r$ denote the two boundary curves of the envelop. Letting λ_r and $\bar{\lambda}_r$ play the roles of $H(r\beta)$ and $H(-r\beta)$ we deform the curve λ into λ_1 by allowing $r \to 0$.

3. **The natural map $\hat{f}$.** It is a characterizing property of continuous normal functions f that for $\eta > 0$ there exists a $\delta = \delta(\eta)$ such that for any z' and z'' in D with $\rho(z', z'') < \delta$ then $\chi(f(z'), f(z'')) < \eta$. This is a direct consequence of the condition that a family of continuous functions is normal in a domain D if and only if it is spherically equicontinuous on compact subsets of D[3, pp. 244–246]. (This was noted by Lappan [5].)

LEMMA 2. *Bet* $A, B \in \Lambda$. *If for each point* $a \in A$ *there exists* $b = b(a) \in B$ *with* $\chi(a,b) < \varepsilon$ *and if for each* $b \in B$ *there exists an* $a = a(b) \in A$ *with* $\chi(a,b) < \varepsilon$ *then* $\operatorname{dist}(A,B) < \varepsilon$.

Proof. This is an obvious consequence of the inequality

$$\chi(a,A) \leqq \chi(a,b(a)) < \varepsilon$$

$$\chi(b,B) \leqq \chi(b,a)) < \varepsilon$$

THEOREM 2. *If f is a normal function then $\hat{f}$ is continuous from Λ into μ'.*

Proof. Let $\lambda_0 \in \Lambda$ and $\varepsilon > 0$. By the normality of f there exists $0 < \delta < 1$ such that if $z', z'' \in D$ with $\rho(z', z'') < \delta$ then $\chi(f(z'), f(z'')) < \varepsilon/2$. Let $a \in \hat{f}(\lambda_0)$ and $\{z_m\}$, $z_m \in \lambda_0, \lim_{m\to\infty} z_m = 1$ and $\lim_{m\to\infty} f(z_m) = a$. Let λ be any point of Λ which is in the $\hat{\rho}$ sphere with center λ_0 and radius δ. About each point z_m construct the non-Euclidean disk D_m with center z_m and radius δ. Choose a sequence of points $\{z'_m\}$ where $z'_m \in D_m \cap \lambda$. There is a subsequence z'_{m_k} with $\lim_{k\to\infty} z_{m_k} = 1$ and $\lim_{k\to\infty} f(z'_{m_k}) = b \in \hat{f}(\lambda)$.

Choosing the associated z_{m_k} we have $\rho(z_{m_k}, z'_{m_k}) < \delta$ and referring to the above $\chi(f(z_{m_k}), f(z_{m_k}))) < \varepsilon/2$. Passing to the limit we have $\chi(a,b) \leqq \varepsilon/2$. Interchanging the role of $\{z_n\}$ and $\{z'_n\}$ and referring to Lemma 2 we have that

$$\operatorname{dist}(\hat{f}(\lambda_0), \hat{f}(\lambda)) < \varepsilon.$$

Thus the sphere $S(\lambda_0, \delta)$ is mapped into the sphere $S(\hat{f}(\lambda_0), \varepsilon)$ which is the theorem.

4. The Lindelöf property. Lehto and Virtanen [6, pp 49–52] have proven the following theorem.

THEOREM. *Let $f(z)$ be meromorphic in D and have asymptotic value α at a point $e^{i\theta} \in C$. If f has not angular limit at $e^{i\theta}$, there is a Jordan curve γ_0 such that for any $\varepsilon > 0$, there is an associated Jordan curve γ_ε in D, terminating at $e^{i\theta}$, with $\rho(\gamma_0, \gamma_\varepsilon) < \varepsilon$, such that f tends to α on γ_ε but not on γ_0.*

NOTE: This is not the exact statement of the theorem of Lehto and Virtanen but is a restatement of the theorem in our notation. Hence we have

THEOREM 3. *If f is meromorphic in D and the function $\hat{f}$ is continuous then f has the Lindelöf property at $e^{i\theta}$.*

Proof. If f does not have the Lindelöf property then there is a continuous curve such that f has limit α on γ but does not have angular limit. The result of Lehto and Virtanen clearly imply that f is not continuous. We might note that a direct application of a theorem of Seidel and Bagemihl [1, p 266] gives that for normal function f, if $\hat{f}(\lambda_1) = \{a\}$ for some $\lambda_1 \in \Lambda$, then $\hat{f} \equiv \{a\}$ on the subspace $\hat{\rho}(\lambda_1, \lambda) < 1$. If we partial order the elements of μ' by set inclusion we then have that if $\hat{f}$ is "smallest" for some value it is constant in some neighborhood.

It would be nice to have a type of maximum property, to wit, if $\hat{f}(\lambda_1)=W$ then $\hat{f}$ is constant for $\hat{\rho}(\lambda_1,\lambda)<1$. But this is not so. The elliptic modular function is the counter-example. Details can be found in [7, p. 262].

We might also note that the set of right (and left) horocycles at a point $e^{i\theta}$ all lie inside a $\hat{\rho}$ disk of radius 1. Bagemihl has recently investigated the behavior of $\hat{f}$ on these disks [1], investigating such problems as under what conditions $\hat{f}\equiv\{a\}$ on the $\hat{\rho}$-disks of left and right horocycles.

Lehto and Virtanen [6, pp. 54–56] have shown that a necessary and sufficient condition that a function f be normal is that $\rho(f)(z)<C/(1-|z|)$, where C is a finite constant and $\rho(f)$ is the spherical derivative of f. The spherical derivative also enables us to establish a sufficient condition that $\chi(f(z'),f(z''))$ shall be arbitrarily small.

We now proceed to Lemma 3.

LEMMA 3. *Let* $\rho(f)(z)=o\,(1/1-|z|)$ *for* $z\in D$, *and* $\{z_m\}$, $\{z_m'\}$ *two sequences* D, $\lim_{m\to\infty}|z_m'|=\lim_{m\to\infty}|z_m|=1$, $\rho(z_m',z_m)<K, m=1,2,\cdots$ *then* $\chi(f(z_m),f(z_m'))$ *tends to zero as m tends to infinity.*

Proof. Assume $\{z_m\}$ and $\{z_m'\}$ are two sequences in D with the properties stated in the theorem. Construct a sequence of non-Euclidean disks $\{N(z_m,K)\}$ with centers z_m and radius K. We know there is a Euclidean disk $D(\zeta_m,(1-|\zeta_m|)t_m)$ with center ζ_m and radius $(1-|\zeta_m|t_m)(0<t_m<1)$ which coincides as a point set with $N(z_m,K)$. For each m let R_m be the rectilinear segment joining z_m to z_m' and C_m the image of R_m under f.

Now

$$\chi(f(z_m),f(z_m'),)\leqq\frac{1}{2}\int_{C_m'}\frac{|dw|}{1+|w^2|}$$

where C_m' is the projection of the great circle joining $w_m=f(z_m)$ to $w_m'=f(z_m')$. By definition of C_m and C_m'

$$\chi(f(z_m),f(z_m'))\leqq\frac{1}{2}\int_{C_m}\frac{|dw|}{1+|w^2|}=\frac{1}{2}\int_{R_m}\frac{|f'(z)|\,|dz|}{1+|f(z)|^2}$$

The condition on the spherical derivative that $\rho(f)(z)=o(1/1-|z|)$ is equivalent to the statement that

$$\rho(f)(z)\leqq\frac{A_r}{1-r},\quad |z|\leqq r,\ \lim_{r\to 1}A_r=0.$$

If $r_m=|\zeta_m|+(1-|\zeta_m|)t_m$ then

$$\chi(f(z_m),f(z_m'))\leqq\frac{1}{2}\,\frac{A_{r_m}}{1-r_m}\int_{R_m}|dz|\leqq\frac{A_{r_m}(1-|\zeta_m|)t_m}{(1-r_m)}.$$

It is easy to show that $\lim_{m\to\infty} t_m = 2L/1 + L^2$, $L = e^{2K} - 1/e^{2K} + 1$. (For details see [9].)
From this result and the equality

$$1 - r_m = (1 - |\zeta_m|)(1 - t_m)$$

we have

$$\chi(f(z_m), \quad f(z'_m)) \leqq \frac{A_{r_m} t_m}{(1 - t_m)}.$$

In the limit $t_m/1 - t_m$ is bounded so that $\lim \chi(f(z_m), f(z'_m)) = 0$ With this lemma we now state

THEOREM 4. *Given $f(z)$ defined in D such that $\rho(f)(z) = o\,(1/1 - |z|)$. Then $\hat{f}(\gamma^*) = \hat{f}(\gamma)$ for all γ^* such that $\rho(\gamma^*, \gamma) < 1$, i.e. $\hat{f}$ is constant on each disk of radius one.*

Proof. For a value $\alpha \in C_\gamma(f)$ there is a sequence $\{z_m\}$, $z_m \in \gamma$, $|z_m| \to 1$ with $f(z_m) \to \alpha$. Since $\hat{\rho}(\gamma^*, \gamma) < 1$ implies $\rho^*(\gamma^*, \gamma) < +\infty$ there is a corresponding sequence $\{z'_m\}$, $z'_m \in \gamma$, $|z'_m| \to 1$ and $\rho(z_m, z'_m) < K$ for all m. We infer then by Lemma 3 that $\alpha \in C_\gamma(f)$. The symmetry of the argument implies the result.

As an example of a holomorphic function $f(z)$ satisfying $\rho(f)(z) = o\,(1/1 - |z|)$ we may consider a spiral domain bounded by Jordan curves $\lambda_1(t)$ and $\lambda_2(t)$ which are spirals in D tending to C with $\lambda_1(0) = \lambda_2(0) = 0$ but otherwise disjoint. Parametrize λ_1 and λ_2 so that $\lambda_1(t) = r_1(t)e^{i\theta(t)}$, $\lambda_2(t) = r_2(t)e^{i\theta(t)}$ where $r_1(t) < r_2(t)$ and $\lim_{t\to 1} r_1(t) = \lim_{t\to 1} r_2(t) = 1$. If Δ is the simply connected region bounded by λ_1 and λ_2 then by Riemann mapping theorem there is a univalent function f mapping D onto Δ.

A result of Seidel and Walsh [10, p 124] is that

$$|f'(z_0)|(1 - |z_0|) \leqq 4D_1(w_0)$$

where $D_1(w_0)$ is the radius of univalence of f^{-1} at $w_0 = f(z_0)$. For any sequence $\{z_m\} \in D$ with $|z_m| \to 1$ we note $D_1(w_m) = D_1(f(z_m)) \to 0$. This implies $\rho(f)(z) = 0(1/1 - |z|)$.

From the theory of prime ends it is clear that for this function f there is a point $e^{i\theta}$ such that $\hat{f}(\tau) = C$ for every path ending at $e^{i\theta}$.

There is a further condition under which Theorem 4 also holds. The notation $R(f, e^{i\theta})$ is used for the range of f where

$$R(f, e^{i\theta}) = \{w \in W |\text{ there is } \{z_m\}, z_m \in D, z_m \to e^{i\theta}, m \to \infty \text{ and } f(z_m) = w\}.$$

THEOREM 5. *If f is a meromorphic function in D such that interior $R(f, e^{i\theta}) = \varnothing$ then given any curve γ we have $\hat{f}(\gamma') = \hat{f}(\gamma)$ for all curves $\gamma' \in \Lambda$ such that $\hat{\rho}(\gamma', \gamma) < 1$.*

Proof. We refer the reader to a paper of Rung [8, pp 48–49] which proves the result in the case of curves $\gamma \in \Lambda^*(\theta)$ and note that the argument is easily extended to cover the other cases.

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